2.6. CONTRACTION

To define contraction of a mixed tensor and show that in each process of contraction, the rank of tensor is reduced by two.

(Vikram University Ujjain 1995; Purvanchal 1997, 96, 91; Kanpur M. Sc. 1999, 98)

If we set in a tensor one covariant and one contravariant suffixes equal, the process is called contraction. We illustrate this by the following example.

Let A^{pq}_{rst} be a tensor. By tensor law of transformation,

$$A'^{pq}_{rst} = A^{ab}_{cdl} \frac{\partial x'^p}{\partial x^a} \frac{\partial x'^q}{\partial x^b} \frac{\partial x^c}{\partial x'^r} \frac{\partial x^d}{\partial x'^s} \frac{\partial x^l}{\partial x'^t}$$

Contracting w.r.t. p and t, i.e., putting $p = t$, we get

$$A'^{pq}_{rsp} = A^{ab}_{cdl} \frac{\partial x'^p}{\partial x^a} \frac{\partial x'^q}{\partial x^b} \frac{\partial x^c}{\partial x'^r} \frac{\partial x^d}{\partial x'^s} \frac{\partial x^l}{\partial x'^p}$$

$$= A^{ab}_{cdl} \left(\frac{\partial x'^p}{\partial x^a} \frac{\partial x^l}{\partial x'^p} \right) \left(\frac{\partial x'^q}{\partial x^b} \frac{\partial x^d}{\partial x'^s} \right) \frac{\partial x^c}{\partial x'^r}$$

$$= A^{ab}_{cdl} \left(\frac{\partial x^l}{\partial x^a} \right) \frac{\partial x'^q}{\partial x^b} \cdot \frac{\partial x^c}{\partial x'^r} \cdot \frac{\partial x^d}{\partial x'^s}$$

$$= A^{ab}_{cdl} \, \delta^l_a \frac{\partial x'^q}{\partial x^b} \cdot \frac{\partial x^c}{\partial x'^r} \cdot \frac{\partial x^d}{\partial x'^s} = A^{ab}_{cda} \frac{\partial x'^q}{\partial x^b} \cdot \frac{\partial x^c}{\partial x' x^r} \cdot \frac{\partial x^d}{\partial x'^s}$$

or, $$A'^{pq}_{rsp} = A^{ab}_{cda} \frac{\partial x'^q}{\partial x^b} \cdot \frac{\partial x^c}{\partial x'^r} \cdot \frac{\partial x^d}{\partial x'^s} \qquad \ldots (1)$$

This proves that A^{pq}_{rsp} is also a tensor and its rank = 3.

But (1) is a law of transformation of tensor of rank 3. Hence A^{pq}_{rsp} is tensor of rank 3 and type (1, 2) while A^{pq}_{rst} is a tensor of rank 5 and type (2, 3). *It means that contraction reduces rank of tensor by two.* Also, we can say that contraction enables us to obtain a tensor of order $r-2$ from a mixed tensor of order r.

Consequently contraction of a tensor of type (r, s) becomes tensor of type $(r-1, s-1)$. ***(Kanpur 1991, 89)***

Remark : Contracting A^{ij}_{pqr}, we get the following different tensors of the same rank :

$$A^{ij}_{iqr}, A^{ij}_{pqi}, A^{ij}_{pir}, A^{ij}_{iqr}, A^{ij}_{pqj}.$$

Similar Problem 1. Let A^{pq}_{rst} be a tensor, choose $p = t$ and $q = s$ and show by contracting that A^{pq}_{rqp} is also a tensor. What is its rank ?

(Kanpur B. Sc. 1996)

Hint. See Article 2.6.

Here put $p = t$, $q = s$ and show that reduced tensor has rank one.

Similar Problem 2. Prove that contraction of a mixed tensor A^i_j is a scalar invariant. ***(Kanpur M. Sc. 1996)***

Solution. Since A^i_j is a tensor and so by tensor law of transformation,

$$A'^i_j = A^\alpha_\beta \frac{\partial x^\beta}{\partial x'^j} \frac{\partial x'^i}{\partial x^\alpha}$$

Contracting w.r.t. i & j, we get $A'^i_i = A^\alpha_\beta \dfrac{\partial x^\beta}{\partial x'^i} \dfrac{\partial x'^i}{\partial x^\alpha} = A^\alpha_\beta \delta^\beta_\alpha = A^\alpha_\alpha = A^i_i$

or, $$A'^i_i = A^i_i$$

$\therefore$ Contraction of A^i_j, *i.e.*, A^i_i is a scalar.

2.7. SYMMETRIC TENSOR

(Kanpur Riemannian Geom. 1982, 79; Meerut 92; Agra 77; Banaras Riemannian Geom. 70)

Definition : A tensor A_{ij} is said to be symmetric if $A_{ij} = A_{ji}$. Similarly a tensor A_{ijk} is symmetric in the suffixes j and k if

$$A_{ijk} = A_{ikj}.$$

Claim 1. *Symmetric property remains unchanged by tensor law of transformation, i.e., if a tensor A_{ij} is symmetric in one coordinate system $x^1, x^2, \ldots, x^n$, then A_{ij} is also symmetric in another co-ordinate system $x'^1, x'^2, \ldots x'^n$.* ***(Kanpur B. Sc. 1994, M. Sc. 2000, 1992)***

Proof. For this we have to show that $A'_{ij} = A'_{ji}$.

$$A'_{ij} = A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^i} \frac{\partial x^\beta}{\partial x'^j} = A_{\beta\alpha} \frac{\partial x^\alpha}{\partial x'^i} \frac{\partial x^\beta}{\partial x'^j}$$

[For A_{ij} is symmetric in i and j and so $A_{\alpha\beta} = A_{\beta\alpha}$]

$$= A_{\beta\alpha} \frac{\partial x^\beta}{\partial x'^j} \frac{\partial x^\alpha}{\partial x'^i} = A'_{ji} \quad \text{or} \quad A'_{ij} = A'_{ji}.$$

Claim 2. *A symmetric tensor A_{ij} has $\frac{n}{2}(n+1)$ independent components in V_n.* *(Kanpur Riemannian Geom. 2000, B. Sc. 2004, Agra 76)*

Proof. A_{ij} is a second rank tensor and hence it has n^2 components in V_n. These components are :

$$\begin{matrix} A_{11} & A_{12} & A_{13} & \ldots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \ldots & A_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n1} & A_{n2} & A_{n3} & \ldots & A_{nn}. \end{matrix}$$

Number of independent components corresponding to a repeated suffix is n.

Number of components corresponding to distinct suffixes is $n^2 - n$. Due to symmetry property this number is reduced to $\frac{n^2-n}{2}$.

Total number of independent components is

$$\frac{n^2-n}{2} + n = \frac{n^2-n+2n}{2} = \frac{n^2+n}{2} = \frac{n}{2}(n+1).$$

Remark 1. A tensor A_{ijk}, symmetric in suffixes i and j, has $\frac{n}{2}(n+1) \cdot n$ $= \frac{n^2}{2}(n+1)$ independent components.

Ex. Define a symmetric and skew-symmetric tensors of the type $(r, 0)$. *(Kanpur Riemannian Geom. 72)*

Solution. Any tensor $A^{i_1 i_2 \cdots i_r}$ is a tensor of the type $(r, 0)$. The tensor $A^{i_1 i_2 \cdots i_r}$ is said to be symmetric w.r.t. the first two suffixes if $A^{i_1 i_2 \cdots i_r} = A^{i_2 i_1 i_3 \cdots i_r}$

and skew symmetric w.r.t. the first two suffixes if

$$A^{i_1 i_2 \cdots i_r} = -A^{i_2 i_1 i_3 \cdots i_r}.$$

Ex. Show that the number of independent components of the metric tensor g_{ij} cannot exceed $\frac{1}{2} n (n+1)$. *(Kanpur Riemannian Geom. 72)*

Solution. The tensor g_{ij} is symmetric (Refer Theorem 1, Chapter 3). Now write proof of claim 2 of the above article 2.7 and there replace A_{ij} by g_{ij}.

2.8. ANTI-SYMMETRIC TENSOR *(Kanpur Riemannian Geom. 1982, 70; Meerut 93; Banaras Riemannian Geom. 69; Agra 77)*

Definition. A tensor A^{ij} is said to be anti-symmetric (or *skew-symmetric*) if $A^{ij} = -A^{ji}$.

Claim 1. *An anti-symmetric tensor A^{ij} has $\frac{n}{2}(n-1)$ independent components.* *(Kanpur Riemannian Geom. 2000, 1999; Agra 78)*

Proof. A^{ij} is second rank tensor and hence it has n^2 components in V_n which are given below :

$$\begin{matrix} A^{11} & A^{12} & A^{13} & \ldots & A^{1n} \\ A^{21} & A^{22} & A^{23} & \ldots & A^{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A^{n1} & A^{n2} & A^{n3} & \ldots & A^{nn} \end{matrix}$$

A^{ij} is anti-symmetric $\Rightarrow A^{ij} = -A^{ji}$ for every i and j.

Taking $i = j$, we get

$$A^{ii} = -A^{ii}$$

$$\Rightarrow \quad 2A^{ii} = 0 \Rightarrow A^{ii} = 0$$

$$\Rightarrow \quad A^{11} = A^{22} = A^{33} = \ldots = A^{nn} = 0.$$

$\Rightarrow$Number of independent components corresponding to a repeated suffix is zero.

Number of components of A^{ij} corresponding to distinct suffixes is $n^2 - n$. Due to anti-symmetry property this number is reduced to $\frac{n^2-n}{2}$.

Total number of independent components of A^{ij}

$$= \frac{n^2-n}{2} + 0 = \frac{n(n-1)}{2}.$$

Deductions. (i) A tensor A_{ijk} is said to be anti-symmetric in suffixes i and j if

$$A^{ijk} = -A^{jik}.$$

This tensor has $\frac{n}{2}(n-1) \cdot n = \frac{n^2}{2}(n-1)$ independent components. *(Kanpur M. Sc. 2002)*

(ii) A tensor A_{ijk} is said to be anti-symmetric in suffixes i, j and k if

$$A_{ijk} = -A_{jik}, \quad \ldots (1)$$
$$A_{ijk} = -A_{kji},$$
$$A_{ijk} = -A_{ikj}.$$

This tensor has $\frac{n}{6}(n-1)(n-2)$ independent components. This is proved as follows :

(a) When A_{ijk} is of the type A_{iii}.

Taking $i = j = k$ in (1), we get $A_{iii} = -A_{iii}$.

This gives $A_{iii} = 0 \; \forall \; i$.

In this case number of independent components of A_{ijk} is zero.

(b) When A_{ijk} is of the type A_{iik},

By anti-symmetry, $A_{iik} = -A_{iik}$.

This $\Rightarrow A_{iik} = 0$.

In this case number of independent components of A_{ijk} is again zero.

It means that the number of indepdendent components of A_{ijk} is

$${}^nC_3 = \frac{n(n-1)(n-2)}{6}.$$

Claim 2. *To show that if a tensor is skew-symmetric with respect to a pair of indices in one system of co-ordinates, it is so in every system.* *(Banaras Riemannian Geom. 1970; Kanpur Riemannian Geom. 70)*

Proof. Let a tensor A_{ij} be anti-symmetric in co-ordinate system $x^1, x^2, \ldots, x^n$ so that

$$A_{ij} = -A_{ji}. \qquad \ldots (1)$$

To prove that A_{ij} is also anti-symmetric in co-ordinate system $x'^1, x'^2, \ldots, x'^n$.

For this we have to show that $A'_{ij} = -A'_{ji}$.

From (1), $\quad A_{\alpha\beta} = -A_{\beta\alpha}$.

$$A'_{ij} = A_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^i}\frac{\partial x^\beta}{\partial x'^j} = -A_{\beta\alpha}\frac{\partial x^\alpha}{\partial x'^i}\frac{\partial x^\beta}{\partial x'^j}$$

$$= -A_{\beta\alpha}\frac{\partial x^\beta}{\partial x'^j}\frac{\partial x^\alpha}{\partial x'^i} = -A'_{ji}.$$

Thus $\quad A'_{ij} = -A'_{ji}$.

Proved.

Problem. ***If A^{ij} and B^{pq} are skew-symmetric tensors, show that their outer product is symmetric tensor.*** *(Kanpur B. Sc. 1995)*

Solution. A^{ij} and B^{pq} are anti-symmetric

$$\Rightarrow \left.\begin{aligned} A^{ij} &= -A^{ji} \\ B^{pq} &= -B^{qp} \end{aligned}\right\} \qquad \ldots (1)$$

Let $\quad C^{ijpq} = A^{ij}B^{pq}$

Then $\quad C^{ijpq} = -A^{ji}(-B^{qp}) = A^{ji}B^{qp} = C^{jiqp}$

This $\Rightarrow C^{ijpq} = A^{ij}B^{pq}$ is symmetric tensor.

Theorem 6. Quotient Law. ***A set of quantities, whose inner product with an arbitrary vector is a tensor, it itself a tensor.***

(Kanpur Riemannian Geom. 2000, 1999, 98, 95, 91; I.A.S. 96; Gorakhpur 96, Purvanchal 96, Avadh 94; Kanpur B. Sc. 1995)

Proof. Let $A^{i_1 i_2 \ldots i_l}_{j_1 j_2 \ldots j_m k}$ be a set of quantities where k, the i's and the j's take the values from 1 to n. Let u^k be an arbitrary vector. Let the inner product

$$A^{i_1 i_2 \ldots i_l}_{j_1 j_2 \ldots j_m k}\, u^k = B^{i_1 i_2 \ldots i_l}_{j_1 j_2 \ldots j_m} \qquad \ldots (1)$$

be a tensor.

To prove that $A^{i_1 i_2 \ldots i_l}_{j_1 j_2 \ldots j_m k}$ is a tensor.

From (1), we get $A^{\alpha_1 \alpha_2 \ldots \alpha_l}_{\beta_1 \beta_2 \ldots \beta_m a}\, u^a = B^{\alpha_1 \alpha_2 \ldots \alpha_l}_{\beta_1 \beta_2 \ldots \beta_m} \qquad \ldots (2)$

and $\quad A'^{i_1 i_2 \ldots i_l}_{j_1 j_2 \ldots j_m k}\, u'^k = B'^{i_1 i_2 \ldots i_l}_{j_1 j_2 \ldots j_m}$

Applying tensor law of transformation, we get

$$A'^{i_1 i_2 \ldots i_l}_{j_1 j_2 \ldots j_m k}\, u'^k = B^{\alpha_1 \alpha_2 \ldots \alpha_l}_{\beta_1 \beta_2 \ldots \beta_m}\frac{\partial x'^{i_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{i_l}}{\partial x^{\alpha_l}}\frac{\partial x^{\beta_1}}{\partial x'^{j_1}}\cdots\frac{\partial x^{\beta_m}}{\partial x'^{j_m}}$$

$$= A^{\alpha_1 \alpha_2 \ldots \alpha_l}_{\beta_1 \beta_2 \ldots \beta_m a}\, u^a\frac{\partial x'^{i_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{i_l}}{\partial x^{\alpha_l}}\frac{\partial x^{\beta_1}}{\partial x'^{j_1}}\cdots\frac{\partial x^{\beta_m}}{\partial x'^{j_m}}, \text{ by (2).}$$

or

$$= A^{\alpha_1 \alpha_2 \ldots \alpha_l}_{\beta_1 \beta_2 \ldots \beta_m a}\, u'^k\frac{\partial x^a}{\partial x'^k}\frac{\partial x'^{i_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{i_l}}{\partial x^{\alpha_l}}\frac{\partial x^{\beta_1}}{\partial x'^{j_1}}\cdots\frac{\partial x^{\beta_m}}{\partial x'^{j_m}}$$

$$u'^k\left(A'^{i_1 \ldots i_l}_{j_1 j_2 \ldots j_m k} - A^{\alpha_1 \ldots \alpha_l}_{\beta_1 \ldots \beta_m a}\frac{\partial x^a}{\partial x'^k}\frac{\partial x'^{i_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{i_l}}{\partial x^{\alpha_l}}\frac{\partial x^{\beta_1}}{\partial x'^{j_1}}\cdots\frac{\partial x^{\beta_m}}{\partial x'^{j_m}}\right) = 0$$

Since u'^k is an arbitrary vector and hence $u'^k \neq 0$.

Therefore the expression within the bracket vanishes. Consequently

$$A'^{i_1 i_2 \ldots i_l}_{j_1 j_2 \ldots j_m k} = A^{\alpha_1 \alpha_2 \ldots \alpha_l}_{\beta_1 \beta_2 \ldots \beta_m a}\frac{\partial x^a}{\partial x'^k}\frac{\partial x'^{i_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{i_l}}{\partial x^{\alpha_l}}\frac{\partial x^{\beta_1}}{\partial x'^{j_1}}\cdots\frac{\partial x^{\beta_m}}{\partial x'^{j_m}}.$$

This confirms the tensor law of transformation. Hence the required result follows.

Remark. A generalised form of the quotient law may be stated as:

A set of quantities, whose inner product with an arbitrary tensor is a tensor, is itself a tensor.

Ex. *State and prove Quotient law of tensors.* *(Kanpur 1984, 78)*

Theorem 7. ***To show that Kronecker delta is a mixed tensor of rank two, by using Quotient law of tensor.***

(Kanpur Riemannian Geom. 1995, 92; B. Sc. 98; Avadh 1993)

Also show that it is invariant, i.e. it has the same components in every co-ordinate system. *(Kanpur B.Sc. 2004)*

Proof. (i) To show that δ^i_j is a mixed tensor of rank two.

Let u^k be an arbitrary vector.

Evidently $\delta^i_j u^j = u^i$ = contravariant tensor of rank one.

Thus the inner product of δ^i_j with an arbitrary vector is a tensor. Hence, by Quotient law, δ^i_j is a tensor. Again δ^i_j has two distinct indices one of which is upper and the other is lower. Hence δ^i_j is a mixed tensor of rank 2.

(ii) To show that δ^i_j has that same components is every co-ordinate system, it is enough to show that $\delta'^i_j = \delta^i_j$.

$$\delta'^i_j = \delta^\alpha_\beta\frac{\partial x'^i}{\partial x^\alpha}\cdot\frac{\partial x^\beta}{\partial x'^j} = \left(\delta^\alpha_\beta\frac{\partial x'^i}{\partial x^\alpha}\right)\frac{\partial x^\beta}{\partial x'^j} = \frac{\partial x'^i}{\partial x^\beta}\cdot\frac{\partial x^\beta}{\partial x'^j} = \frac{\partial x'^i}{\partial x'^j} = \delta^i_j.$$

$\therefore \quad \delta'^i_j = \delta^i_j$.

Proved.

Theorem 8. ***To prove that if the components of a tensor vanishes in one co-ordinate system, they vanish identically in all co-ordinate systems.*** *(Kanpur Riemannian Geom. 2000, Meerut 93)*

Solution. A^i_j is a tensor such that it vanishes in one system (x^i-system) and so

$$A^i_j = 0 \;\forall\; i \text{ and } j \qquad \ldots (1)$$

To prove that $A'^i_j = 0$ in every system. By tensor law of transformation

$$A'^i_j = A^\alpha_\beta\frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^j} \qquad \ldots (2)$$

By (1), $\quad A^\alpha_\beta = 0$. Put this in (2), $A'^i_j = 0$. $\qquad \ldots (3)$

Again by tensor law of transformation,

$$A''^i_j = A'^\alpha_\beta\frac{\partial x''^i}{\partial x'^\alpha}\frac{\partial x'^\beta}{\partial x''^j} \qquad \ldots (4)$$

By (3), $A'^{\alpha}_{\beta} = 0$. Putting this in (4), we get $A''^{i}_{j} = 0$.

Thus we have proved that

$$A^i_j = 0 \Rightarrow A'^i_j = 0 \Rightarrow A''^i_j = 0 \text{ etc.}$$

This proves that if a tensor vanishes in one system, it vanishes in every system.

2.9. RECIPROCAL (OR CONJUGATE) SYMMETRIC TENSOR

Theorem. *If a_{ij} be the components of a symmetric tensor of the type (0, 2), such that $a \overset{def}{=} | a_{ij} | \neq 0$, prove that it is possible to define a tensor whose components a^{ij} satisfy $a^{ij} a_{jk} = \delta^i_k$. Prove further that this tensor is a contravariant symmetric tensor of second order and $a\, a^{ij}$ = cofactor of a_{ij} in a.*

(Kanpur Riemannian Geom. 1982, 74, 86, Meerut 93)

Proof. Consider a covariant symmetric tensor a_{ij} of second order. Denote by a the determinant $| a_{ij} |$, *i.e.*, $a = | a_{ij} |$.

We also denote the cofactor of a_{ij} in the determinant $| a_{ij} |$ by A^{ji}. The normalised cofacotr of a_{ij} in the determinant $| a_{ij} |$ is denoted by $\frac{1}{a} A^{ji}$. We define

$$a^{ij} = \frac{\text{cofactor of } a_{ij} \text{ in the determinant } | a_{ij} |}{a}$$

$$= \frac{A^{ji}}{a}, \text{ by our assumption}$$

or

$$a^{ij} = \frac{A^{ji}}{a}.$$

This declares that a^{ij} is the normalised cofactor of a_{ij} in $| a_{ij} |$. Now we shall show that a^{ij} is a second rank contravariant symmetric tensor.

$$a_{ij} \text{ is symmetric} \Rightarrow | a_{ij} | \text{ is symmetric}$$

$$\Rightarrow A^{ji} \text{ is symmetric}$$

$$\Rightarrow \frac{1}{a} A^{ji} \text{ is symmetric}$$

$$\Rightarrow a^{ij} \text{ is symmetric.}$$

Let u^i be an arbitrary vector. Since the product of two tensors is a tensor and hence $u^i a_{ij}$ is a tensor.

Write $$B_j = u^i a_{ij}.$$

Now B_j is an arbitrary vector due to the fact that u^j is arbitrary.

$$B_j a^{jk} = u^i a_{ij} a^{jk} = u^i a_{ij} \frac{A^{kj}}{a}$$

$$= \frac{u^i a \delta^k_i}{a}, \text{ by a well known property of determinant.}$$

$$= u^i \delta^k_i = u^k = \text{a contravariant tensor of rank one}$$

$$\therefore \quad B_j a^{jk} = \text{tensor.}$$

This proves by Quotient law that a^{jk} is a tensor of the type as indicated by its suffixes. Hence a^{ij} is a contravariant second order tensor. We have already shown that a^{ij} is symmetric. Finally, a^{ij} is contravariant second order symmetric tensor.

The tensors a_{ij} and a^{ij} are defined as reciprocal to each other. They are also sometimes called conjugate tensors.

Remark. (i) $| a^{ij} | = \dfrac{1}{a}$. (See problem 5, page 39)

(ii) $a^{ij} a_{jk} = \delta^i_k$.

2.10. INVARIANT

Any function I of co-ordinates x^i is called an invariant or **scalar** if $I' = I$, where I' is the value of I in new co-ordinate system x'^i. For example $A^i B_i$ and δ^i_j are invariant quantities.

2.11. RELATIVE TENSOR

If the quantities A_{ij} satisfy the following transformation law

$$A'_{ij} = A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^i} \cdot \frac{\partial x^\beta}{\partial x'^j} \left| \frac{\partial x}{\partial x'} \right|^\omega$$

then A_{ij} is called a **relative tensor of weight** ω. A relative tensor of weight one is called **tensor density**. If the weight of a relative tensor is zero, then the relative tensor is called an **absolute tensor.**

2.12. RELATIVE VECTOR

A relative tensor of rank one is called *relative vector*. Thus if

$$A'_i = A_\alpha \frac{\partial x^\alpha}{\partial x'^i} \left| \frac{\partial x}{\partial x'} \right|^\omega$$

then A_i is called a relative vector of weight ω. A relative vector of weight one is called **vector density**. A relative vector of weight zero is called **absolute vector.**